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Closed string amplitudes as single-valued open string amplitudes

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Abstract

We show that the single trace heterotic N -point tree-level gauge amplitude $\mathcal{A}_N^{\text{HET}}$ can be obtained from the corresponding type I amplitude \mathcal{A}_N^{I} by the single-valued (sv) projection: $\mathcal{A}_N^{\text{HET}} = \text{sv}(\mathcal{A}_N^{\text{I}})$. This projection maps multiple zeta values to single-valued multiple zeta values. The latter represent a subclass of multiple zeta values originating from single-valued multiple polylogarithms at unity. Similar relations between open and closed string amplitudes or amplitudes of different string vacua can be established. As a consequence the α' -expansion of a closed string amplitude is dictated by that of the corresponding open string amplitude. The combination of single-valued projections, Kawai–Lewellen–Tye relations and Mellin correspondence reveal a unity of all tree-level open and closed superstring amplitudes together with the maximally supersymmetric Yang–Mills and supergravity theories.

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1. Introduction

Perturbative open and closed string amplitudes seem to be rather different due to their underlying different world-sheet topologies. On the other hand, mathematical methods entering their computations reveal some unexpected connections. For example, at the string tree-level, the Kawai–Lewellen–Tye (KLT) relations expose a non-trivial connection between open and closed

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superstring amplitudes [1]. Similarly, tree-level scattering involving both open and closed strings can entirely be described by open string amplitudes only [2]. Furthermore, recently in Ref. [3] a new relation between open (type I) and closed (type II) superstring amplitudes has been found. The α' -expansion (with α' being the inverse string tension) of the closed superstring amplitude can be cast into the same algebraic form as the open superstring amplitude: the closed superstring amplitude is essentially the single-valued (sv) version of the open superstring amplitude.

All these encountered string properties of scattering amplitudes suggest that there is a deeper connection between perturbative gauge and gravity theories than expected. Furthermore, these observations in string theory are attended by results in field theory. There an apparent similarity between perturbative gauge- and gravity-theories is established through the double copy construction [4,5]. The latter can indeed be furnished by considering gauge amplitudes in heterotic string theory as a simple consequence of their underlying world-sheet structure [6,7]. Besides, recently in Ref. [8] an interesting uniform description of field-theoretical gauge- and gravity amplitudes has been presented and we shall see that some of its underlying structure is carried by the gauge amplitudes in heterotic string theory.

Yang–Mills (YMs) and supergravity amplitudes appear in the $\alpha' \rightarrow 0$ limit of superstring theory. The recently discovered Mellin correspondence [9] allows (re)constructing the full-fledged tree-level open superstring amplitudes (to all orders in α') from this field-theory limit. In this work, we employ the single-valued projection as a link between type I and heterotic string theory. Together with the sv connection between type I and type II, Mellin correspondence and KLT relations, it appears in the web of connections between full-fledged string and field-theoretical tree-level amplitudes. The basic building blocks of superstring amplitudes are supplied by gauge theory describing the string zero modes.

The paper is organized as follows. In Section 2 we review the subspace of single-valued multiple zeta values, in which a large class of complex sphere integrals of closed superstring amplitudes lives. After exhibiting results on the tree-level open superstring N -point disk amplitude in Section 3 we compute the analogous heterotic single trace N -gauge boson amplitude in a generic heterotic string vacuum. We shall find, that the two amplitudes are related by the sv projection. As a byproduct we verify that heterotic string subamplitudes fulfill the same amplitude relations, *i.e.* Kleiss–Kuijff (KK) and Bern–Carrasco–Johansson (BCJ) relations as field-theory subamplitudes. In Section 4 we discuss closed superstring amplitudes involving gravitons or scalars and relate them to open superstring amplitudes. In Section 5 we elucidate the generic basic building blocks of open and closed string world-sheet integrals and establish a relation between them. This correlation is responsible for the connection between open and closed superstring amplitudes. In Section 6, we discuss connections between string and field-theoretical amplitudes and comment on their relation to the web of string dualities.

2. Single-valued multiple zeta-values in superstring theory

The analytic dependence on the inverse string tension α' of string tree-level amplitudes furnishes an extensive and rich mathematical structure, which is related to modern developments in number theory and arithmetic algebraic geometry, *cf.* Refs. [10,3].

The topology of the string world-sheet describing tree-level scattering of open strings is a disk, while tree-level scattering of closed strings is characterized by a complex sphere. Open string amplitudes are expressed by integrals along the boundary of the world-sheet disk (real projective line) as iterated (real) integrals on $\mathbf{RP}^1 \setminus \{0, 1, \infty\}$, whose values (more precisely the coefficients in their power series expansion in α') are given by multiple zeta values (MZVs)

$$\zeta_{n_1, \dots, n_r} := \zeta(n_1, \dots, n_r) = \sum_{0 < k_1 < \dots < k_r} \prod_{l=1}^r k_l^{-n_l}, \quad n_l \in \mathbf{N}^+, \quad n_r \geq 2, \quad (2.1)$$

with r specifying the depth and $w = \sum_{l=1}^r n_l$ denoting the weight of the MZV ζ_{n_1, \dots, n_r} . On the other hand, closed string amplitudes are given by integrals over the complex world-sheet sphere as iterated integrals on $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ integrated independently on all choices of paths. While in the α' -expansion of open superstring tree-level amplitudes generically the whole space of MZVs (2.1) enters [11,10], closed superstring tree-level amplitudes exhibit only a subset of MZVs appearing in their α' -expansion [11,10]. This subclass can be identified [3] as single-valued multiple zeta values (SVMZVs)

$$\zeta_{\text{sv}}(n_1, \dots, n_r) \in \mathbf{R} \quad (2.2)$$

originating from single-valued multiple polylogarithms (SVMPs) at unity [12]. SVMZVs have recently been studied by Brown in [13] from a mathematical point of view. They have been identified as the coefficients in an infinite series expansion of the Deligne associator [14] in two non-commutative variables. On the other hand, from a physical point of view SVMZVs appear in the computation of graphical functions for certain Feynman amplitudes [15].

The numbers (2.2) can be obtained from the MZVs (2.1) by introducing the following homomorphism:

$$\text{sv} : \zeta_{n_1, \dots, n_r} \mapsto \zeta_{\text{sv}}(n_1, \dots, n_r). \quad (2.3)$$

The numbers (2.2) satisfy the same double shuffle and associator relations than the usual MZVs (2.1) and many more relations [13]. For instance we have (cf. Ref. [3] for more examples):

$$\begin{aligned} \text{sv}(\zeta_2) &= \zeta_{\text{sv}}(2) = 0, \\ \text{sv}(\zeta_{2n+1}) &= \zeta_{\text{sv}}(2n+1) = 2\zeta_{2n+1}, \quad n \geq 1, \\ \text{sv}(\zeta_{3,5}) &= -10\zeta_3\zeta_5, \quad \text{sv}(\zeta_{3,7}) = -28\zeta_3\zeta_7 - 12\zeta_5^2, \\ \text{sv}(\zeta_{3,3,5}) &= 2\zeta_{3,3,5} - 5\zeta_3^2\zeta_5 + 90\zeta_2\zeta_9 + \frac{12}{5}\zeta_2^2\zeta_7 - \frac{8}{7}\zeta_2^3\zeta_5^2, \quad \dots \end{aligned} \quad (2.4)$$

Strictly speaking, the map sv is defined in the Hopf algebra \mathcal{H} of motivic MZVs ζ^m . Motivic MZVs¹ ζ^m are defined as elements of the algebra $\mathcal{H} = \bigoplus_{w \geq 0} \mathcal{H}_w$ over \mathbf{Q} , which is graded for the weight w and equipped with the period homomorphism $\text{per} : \mathcal{H} \rightarrow \mathbf{R}$, which maps $\zeta_{n_1, \dots, n_r}^m$ to ζ_{n_1, \dots, n_r} , i.e. $\text{per}(\zeta_{n_1, \dots, n_r}^m) = \zeta_{n_1, \dots, n_r}$ [16]. In the algebra \mathcal{H} the homomorphism $\text{sv} : \mathcal{H} \rightarrow \mathcal{H}^{\text{sv}}$, with

$$\zeta_{n_1, \dots, n_r}^m \mapsto \zeta_{\text{sv}}^m(n_1, \dots, n_r), \quad (2.5)$$

and

$$\zeta_{\text{sv}}^m(2) = 0 \quad (2.6)$$

can be constructed [13]. The motivic SVMZVs $\zeta_{\text{sv}}^m(n_1, \dots, n_r)$ generate the subalgebra \mathcal{H}^{sv} of the Hopf algebra \mathcal{H} and satisfy all motivic relations between MZVs.

¹ Motivic aspects of MZVs have recently matured in describing tree-level open superstring amplitudes in Ref. [10] and tree-level closed superstring amplitudes in Ref. [3].

In practice, the map sv is constructed recursively in the (trivial) algebra-comodule $\mathcal{U} = \mathcal{U}' \otimes_{\mathbf{Q}} \mathbf{Q}[f_2]$ with the first factor $\mathcal{U}' = \mathbf{Q}\langle f_3, f_5, \dots \rangle$ generated by all non-commutative words in the letters f_{2i+1} [17]. We have $\mathcal{H} \simeq \mathcal{U}$, in particular $\zeta_{2i+1} \simeq f_{2i+1}$. The homomorphism

$$\text{sv} : \mathcal{U}' \rightarrow \mathcal{U}', \quad (2.7)$$

with

$$w \mapsto \sum_{uv=w} u \mathbin{\text{\texttt{III}}} \tilde{v}, \quad (2.8)$$

and

$$\text{sv}(f_2) = 0 \quad (2.9)$$

maps (e.g. $\text{sv}(f_{2i+1}) = 2f_{2i+1}$) the algebra of non-commutative words $w \in \mathcal{U}$ to the smaller subalgebra \mathcal{U}^{sv} , which describes the space of SVMZVs [13]. In Eq. (2.7) the word \tilde{v} is the reversal of the word v and $\mathbin{\text{\texttt{III}}}$ is the shuffle product. For more details we refer the reader to the original reference [13] and subsequent applications in [3].

In supersymmetric Yang–Mills (SYM) theory a large class of Feynman integrals in four space–time dimensions lives in the subspace of SVMZVs or SVMs, cf. Refs. [18]. As pointed out by Brown in [13], this fact opens the interesting possibility to replace general amplitudes with their single-valued versions (defined by the map sv), which should lead to considerable simplifications. In string theory this simplification occurs by replacing open superstring amplitudes by their single-valued versions describing closed superstring amplitudes.

3. Heterotic gauge amplitudes as single-valued type I gauge amplitudes

In this section we shall compute single trace N -gluon tree-level subamplitudes in heterotic string vacua. Our results will then be compared with the corresponding type I amplitudes.

Let us first review the latter. The open superstring N -gluon tree-level amplitude $\mathfrak{A}_N^{\text{I}}$ decomposes into a sum

$$\mathfrak{A}_N^{\text{I}} = (g_{\text{YM}}^{\text{I}})^{N-2} \sum_{\Pi \in S_N / \mathbf{Z}_2} \text{Tr}(T^{a_{\Pi(1)}} \dots T^{a_{\Pi(N)}}) \mathcal{A}^{\text{I}}(\Pi(1), \dots, \Pi(N)) \quad (3.1)$$

over color ordered subamplitudes $\mathcal{A}^{\text{I}}(\Pi(1), \dots, \Pi(N))$ supplemented by a group trace in the fundamental representation. Above, the YM coupling is denoted by g_{YM}^{I} , which in type I superstring theory is given by $g_{\text{YM}}^{\text{I}} \sim e^{\Phi/2}$ with the dilaton field Φ . The sum runs over all permutations S_N of labels $i = 1, \dots, N$ modulo cyclic permutations \mathbf{Z}_2 , which preserve the group trace. A vector \mathcal{A}^{I} with its entries $\mathcal{A}_{\pi}^{\text{I}} = \mathcal{A}^{\text{I}}(\pi)$ describing the $(N-3)!$ independent open N -point superstring subamplitudes

$$\mathcal{A}^{\text{I}}(\pi) := \mathcal{A}^{\text{I}}(1, \pi(2), \dots, \pi(N-2), N-1, N), \quad \pi \in S_{N-3} \quad (3.2)$$

can be given by the matrix expression $\mathcal{A}^{\text{I}} = FA$ [19,20]. In components the latter reads

$$\mathcal{A}^{\text{I}}(\pi) = \sum_{\sigma \in S_{N-3}} F_{\pi\sigma} A(\sigma), \quad \pi \in S_{N-3}, \quad (3.3)$$

with the $(N-3)!$ (independent) SYM subamplitudes

$$A(\sigma) := A(1, \sigma(2), \dots, \sigma(N-2), N-1, N), \quad \sigma \in S_{N-3}, \quad (3.4)$$

which constitute the vector A with entries $A_\sigma = A(\sigma)$ and the period matrix $F_{\pi\sigma}$. According to [21] the latter can be expressed in terms of basic open string world-sheet disk integrals ($z_{ij} = z_i - z_j$)

$$Z_\pi(\rho) := Z_\pi(1, \rho(2), \dots, \rho(N-2), N, N-1) \\ = \left(\prod_{j=2}^{N-2} \int_{D(\pi)} dz_j \right) \frac{\prod_{i < j}^{N-1} |z_{ij}|^{s_{ij}}}{z_{1\rho(2)} z_{\rho(2), \rho(3)} \cdots z_{\rho(N-3), \rho(N-2)}} \quad (3.5)$$

as:

$$F_{\pi\sigma} = (-1)^{N-3} \sum_{\rho \in S_{N-3}} Z_\pi(\rho) S[\rho|\sigma]. \quad (3.6)$$

Above we have the KLT kernel² [1,22,23]

$$S[\rho|\sigma] := S[\rho(2), \dots, \rho(N-2) | \sigma(2), \dots, \sigma(N-2)] \\ = \prod_{j=2}^{N-2} \left(s_{1,j_\rho} + \sum_{k=2}^{j-1} \theta(j_\rho, k_\rho) s_{j_\rho, k_\rho} \right), \quad (3.7)$$

with $j_\rho = \rho(j)$ and $\theta(j_\rho, k_\rho) = 1$ if the ordering of the legs j_ρ, k_ρ is the same in both orderings $\rho(2), \dots, \rho(N-2)$ and $\sigma(2), \dots, \sigma(N-2)$, and zero otherwise. Due to conformal invariance on the world-sheet in (3.5) we have fixed the world-sheet positions as $z_1 = 0$, $z_{N-1} = 1$, $z_N = \infty$ and the remaining $N-3$ points z_i are integrated along the boundary of the disk subject to the ordering $D(\pi) = \{z_j \in \mathbf{R} \mid 0 < z_{\pi(2)} < \cdots < z_{\pi(N-2)} < 1\}$. Furthermore, we have the real numbers:

$$s_{ij} = \alpha' (k_i + k_j)^2 = 2\alpha' k_i k_j. \quad (3.8)$$

The latter describe the $\frac{1}{2}N(N-3)$ independent kinematic invariants of the scattering process involving N external momenta k_i , $i = 1, \dots, N$ and α' is the inverse string tension. Note, that in the field-theory limit $\alpha' \rightarrow 0$ we have

$$Z|_{\alpha'=0} = (-1)^{N-3} S^{-1}, \quad \text{i.e.} \quad F|_{\alpha'=0} = 1. \quad (3.9)$$

Let us now move on to the gluon scattering in heterotic string vacua. The string world-sheet describing the tree-level scattering of N closed strings has the topology of a complex sphere with N insertions of vertex operators. The string S -matrix elements can be computed from the expression

$$\mathfrak{A}_N^{\text{HET}} = V_{\text{CKG}}^{-1} \left(\prod_{j=1}^N \int_{z_j \in \mathbf{C}} d^2 z_j \right) \langle V(z_1, \bar{z}_1, \xi_1, k_1) \cdots V(z_N, \bar{z}_N, \xi_N, k_N) \rangle \quad (3.10)$$

involving N gluon vertex operators $V(z_l, \bar{z}_l, \xi_l, k_l)$. In heterotic string vacua in the RNS formalism the latter are given as [24]

² The matrix S with entries $S_{\rho,\sigma} = S[\rho|\sigma]$ is defined as a $(N-3)! \times (N-3)!$ matrix with its rows and columns corresponding to the orderings $\rho \equiv \{\rho(2), \dots, \rho(N-2)\}$ and $\sigma \equiv \{\sigma(2), \dots, \sigma(N-2)\}$, respectively. The matrix S is symmetric, i.e. $S^t = S$.

$$\begin{aligned}
V^{(-1)}(z, \bar{z}, \xi, k) &= g_c \xi_\mu J^a(z) e^{-\phi(\bar{z})} \psi^\mu(\bar{z}) e^{ik_\rho X^\rho(z, \bar{z})}, \\
V^{(0)}(z, \bar{z}, \xi, k) &= g_c \left(\frac{2}{\alpha'} \right)^{1/2} \xi_\mu J^a(z) \left[i \partial X^\mu(\bar{z}) + \frac{\alpha'}{2} k_\lambda \psi^\lambda(\bar{z}) \psi(\bar{z})^\mu \right] e^{ik_\rho X^\rho(z, \bar{z})},
\end{aligned} \tag{3.11}$$

in the (-1) - and zero-ghost picture, respectively. The polarization ξ_μ satisfies the on-shell condition $\xi_\mu k^\mu = 0$, with the external space-time momenta k subject to the on-shell constraint $k_\mu k^\mu = 0$. The heterotic closed string coupling g_c is given by $g_c = (4\pi)^{-1} \alpha'^{1/2} g_{\text{YM}}^{\text{HET}}$, with the heterotic YM coupling $g_{\text{YM}}^{\text{HET}} \sim e^\Phi$. In order to cancel the background ghost charge on the sphere, two vertices in the correlator (3.10) will be inserted in the (-1) -ghost picture, with the remaining ones in the zero-ghost picture. Furthermore, in (3.10), the factor V_{CKG} accounts for the volume of the conformal Killing group of the sphere after choosing the conformal gauge. It will be canceled by fixing three vertex positions and introducing the respective c -ghost correlator.

Since we are discussing tree-level string amplitudes and shall not be concerned with world-sheet modular invariance or finiteness of string loop amplitudes, our findings hold for any space-time dimensions, any amount of supersymmetries ($\mathcal{N} \geq 1$) and any gauge group. In the following we shall discuss the case of a specific heterotic vacuum with gauge group $SO(2n)$ at level one. Then we can write the gauge currents J^a as

$$J^a = \frac{1}{2} (T^a)_{ij} : \psi^i \psi^j :, \quad a = 1, \dots, n(2n-1), \tag{3.12}$$

with $2n$ free fermions ψ^i in a real representation of the Lie group G . The normalization of the representation matrices T^a satisfies³ $\text{tr}(T^a T^b) = 2\delta^{ab}$.

Closed string amplitudes decompose into correlators involving only left-movers and correlators of only right-movers. Hence, the correlator of vertex operators in (3.10) is evaluated by performing all possible Wick contractions in both sectors separately.

In the scattering amplitude (3.10) the gauge vertex operators (3.11) contribute the current correlator $\langle J^{a_1}(z_1) \dots J^{a_N}(z_N) \rangle$ to the left moving sector. This correlator can be evaluated as a sum over permutations $\rho \in S'_N$ without fixed points [25]

$$\langle J^{a_1}(z_1) \dots J^{a_N}(z_N) \rangle = \sum_{\substack{\rho \in S'_N \\ \rho = \xi_1 \dots \xi_r}} (-1)^r \prod_{l=1}^r f_{\xi_l}, \tag{3.13}$$

with each permutation ρ written as product of cycles, $\rho = \xi_1 \dots \xi_r$ and one cycle $\xi = (i_1, i_2, \dots, i_m)$ contributing a factor f_ξ with:

$$f_\xi = \frac{1}{2} \frac{\text{tr}(T^{a_{i_1}} \dots T^{a_{i_m}})}{z_{i_1 i_2} z_{i_2 i_3} \dots z_{i_m i_1}}. \tag{3.14}$$

The number $d(N)$ of derangements S'_N is given by $d(N) = N! \sum_{i=2}^N \frac{(-1)^i}{i!}$, $N \geq 2$. Hence, we have $d(2) = 1$, $d(3) = 2$, $d(4) = 9$, $d(5) = 44$, \dots . E.g. for $N = 4$ we have the nine derangements (2341), (2413), (3142), (3421), (4123), (4312) with $r = 1$ and (2143), (3412), (4321) with $r = 2$, respectively. In (3.13) these nine permutations give rise to [24]:

³ This normalization corresponds to taking the roots to have length two. An explicit representation of the $SO(2n)$ generators T^a in the vector representation is $(T^{kl})_{ij} = \delta_i^k \delta_j^l - \delta_j^k \delta_i^l$. With this the currents become $J^{kl} = : \psi^k \psi^l :$.

$$\langle J^{a_1}(z_1) J^{a_2}(z_2) J^{a_3}(z_3) J^{a_4}(z_4) \rangle = \frac{1}{4} \frac{\text{tr}(T^{a_1} T^{a_2}) \text{tr}(T^{a_3} T^{a_4})}{z_{12}^2 z_{34}^2} - \frac{\text{tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4})}{z_{12} z_{23} z_{34} z_{41}} - (2 \leftrightarrow 3) - (2 \leftrightarrow 4). \quad (3.15)$$

In the following in (3.13) we shall be interested in the single trace (s.t.) contributions of (3.13). The latter appear for $r = 1$, i.e. cycles, which do not factorize into products. Hence the relevant part of (3.13) is:

$$\langle J^{a_1}(z_1) \cdots J^{a_N}(z_N) \rangle_{\text{s.t.}} = -\frac{1}{2} \sum_{\rho \in S_{N-1}} \frac{\text{tr}(T^{a_1} T^{\rho(2)} \cdots T^{\rho(N)})}{z_{1\rho(2)} z_{\rho(2),\rho(3)} \cdots z_{\rho(N-1),\rho(N)} z_{\rho(N),1}}. \quad (3.16)$$

The correlator (3.16) is dressed with the Koba–Nielsen factor $\prod_{i < j}^N |z_{ij}|^{s_{ij}}$ and for a generic group trace $\text{tr}(T^{a_1} T^{\rho(2)} \cdots T^{\rho(N)})$ the total contribution from the left-moving part amounts to

$$\mathcal{A}_L(1, \rho(2), \dots, N) = -\frac{\prod_{i < j}^N |z_{ij}|^{s_{ij}}}{z_{1\rho(2)} z_{\rho(2),\rho(3)} \cdots z_{\rho(N-1),\rho(N)} z_{\rho(N),1}}. \quad (3.17)$$

Alternatively, the single trace gauge correlator (3.16) may be written as sum over $(N-2)!$ permutations as

$$\begin{aligned} & \langle J^{a_1}(z_1) \cdots J^{a_N}(z_N) \rangle_{\text{s.t.}} \\ &= -2^{N-3} \sum_{\sigma \in S_{N-2}} \frac{f^{a_1 a_{\sigma(2)} c_1} f^{c_1 a_{\sigma(3)} c_2} \cdots f^{c_{N-3} a_{\sigma(N)} a_{N-1}}}{z_{1,\sigma(2)} z_{\sigma(2),\sigma(3)} \cdots z_{\sigma(N-2),\sigma(N)} z_{\sigma(N),N-1} z_{N-1,1}}, \end{aligned} \quad (3.18)$$

with the structure constants $f^{abc} = \frac{1}{2} \text{tr}([T^a, T^b] T^c)$. Agreement between the two expressions (3.16) and (3.18) can be shown by using partial fraction decompositions in the positions z_i , $f^{a_1 a_2 c_1} f^{c_1 a_3 c_2} \cdots f^{c_{N-3} a_{N-1} a_N} = 2^{2-N} \text{tr}(T^{a_1} [T^{a_2}, [T^{a_3}, \dots, [T^{a_N}, T^{a_{N-1}}] \dots]])$ and the Jacobi relation $f^{a_1 a_2 c} f^{c a_3 a_4} - f^{a_2 a_3 c} f^{c a_4 a_1} - f^{a_4 a_2 c} f^{c a_3 a_1} = 0$. The form (3.18), which uses a basis of $(N-2)!$ building blocks [26] (cf. also Eq. (3.5)), is suited⁴ to extract BCJ numerators [4] in terms of structure constants.

In (3.10) the right-moving sector of the vertex operators (3.11) constitutes the space–time dependent part of the heterotic amplitude. In fact, this part represents a copy⁵ of the open superstring and we may simply borrow the results from the N -gluon computation presented at the beginning of this section. Hence, before fixing conformal invariance the right-moving part gives rise to:

$$\begin{aligned} \mathcal{A}_R &= (-1)^{N-3} \sum_{\sigma \in S_{N-3}} A(\sigma) \\ &\times \sum_{\bar{\rho} \in S_{N-3}} \frac{\prod_{i < j}^N |\bar{z}_{ij}|^{s_{ij}}}{\bar{z}_{1\bar{\rho}(2)} \bar{z}_{\bar{\rho}(2),\bar{\rho}(3)} \cdots \bar{z}_{\bar{\rho}(N-3),\bar{\rho}(N-2)} \bar{z}_{\bar{\rho}(N-2),N} \bar{z}_{N,N-1} \bar{z}_{N-1,1}} S[\bar{\rho}|\sigma]. \end{aligned} \quad (3.19)$$

⁴ Explicitly, for $N = 4$ we have $\langle J^{a_1}(z_1) \cdots J^{a_4}(z_4) \rangle_{\text{s.t.}} = -2(\frac{f^{a_1 a_2 c} f^{c a_4 a_3}}{z_{12} z_{34}} + \frac{f^{a_1 a_4 c} f^{c a_2 a_3}}{z_{23} z_{41}})$, while for $N = 5$ we obtain $\langle J^{a_1}(z_1) \cdots J^{a_5}(z_5) \rangle_{\text{s.t.}} = -4(\frac{f^{a_1 a_2 c} f^{c a_3 d} f^{d a_5 a_4}}{z_{12} z_{23} z_{45}} + \frac{f^{a_1 a_3 c} f^{c a_2 d} f^{d a_5 a_4}}{z_{13} z_{32} z_{45}} + \frac{f^{a_1 a_3 c} f^{c a_5 d} f^{d a_2 a_4}}{z_{13} z_{24} z_{45}} + \frac{f^{a_1 a_5 c} f^{c a_3 d} f^{d a_2 a_4}}{z_{24} z_{32} z_{45}} + \frac{f^{a_1 a_2 c} f^{c a_5 d} f^{d a_3 a_4}}{z_{12} z_{34} z_{45}} + \frac{f^{a_1 a_5 c} f^{c a_2 d} f^{d a_3 a_4}}{z_{23} z_{34} z_{45}})$ for the gauge choice $z_1 = 0, z_{N-1} = 1, z_N = \infty$.

⁵ The right-moving parts of the vertex operators (3.11) agree with the open superstring gluon vertex operators after the replacement $\alpha' \rightarrow \frac{\alpha'}{4}$. As a consequence in all our subsequent heterotic results this rescaling still has to be performed.

Therefore, after putting together the pieces from the left-movers (3.17) and right-movers (3.19) and taking into account all normalizations the single trace part of (3.10), which constitutes

$$\mathfrak{A}_{N,\text{s.t.}}^{\text{HET}} = (g_{\text{YM}}^{\text{HET}})^{N-2} \sum_{\Pi \in \mathcal{S}_N / \mathbb{Z}_2} \text{tr}(T^{a_{\Pi(1)}} \dots T^{a_{\Pi(N)}}) \mathcal{A}^{\text{HET}}(\Pi(1), \dots, \Pi(N)), \quad (3.20)$$

yields for a generic group trace $\text{tr}(T^{a_1} T^{a_{\rho(2)}} \dots T^{a_{\rho(N-2)}} T^{a_{N-1}} T^{a_N})$ the heterotic subamplitude

$$\begin{aligned} A^{\text{HET}}(\rho) &= V_{\text{CKG}}^{-1} \left(\prod_{j=1}^N \int_{z_j \in \mathbb{C}} d^2 z_j \right) \mathcal{A}_L(1, \rho(2), \dots, N-2, N-1, N) \mathcal{A}_R \\ &= (-1)^{N-3} \sum_{\sigma \in \mathcal{S}_{N-3}} \sum_{\bar{\rho} \in \bar{\mathcal{S}}_{N-3}} J[\rho | \bar{\rho}] S[\bar{\rho} | \sigma] A(\sigma), \end{aligned} \quad (3.21)$$

with the complex integral:

$$\begin{aligned} J[\rho | \bar{\rho}] &:= V_{\text{CKG}}^{-1} \left(\prod_{j=1}^N \int_{z_j \in \mathbb{C}} d^2 z_j \right) \prod_{i < j}^N |z_{ij}|^{2s_{ij}} \frac{1}{z_{1\rho(2)} z_{\rho(2),\rho(3)} \dots z_{\rho(N-2),N-1} z_{N-1,N} z_{N,1}} \\ &\quad \times \frac{1}{\bar{z}_{1\bar{\rho}(2)} \bar{z}_{\bar{\rho}(2),\bar{\rho}(3)} \dots \bar{z}_{\bar{\rho}(N-3),\bar{\rho}(N-2)} \bar{z}_{\bar{\rho}(N-2),N} \bar{z}_{N,N-1} \bar{z}_{N-1,1}}. \end{aligned} \quad (3.22)$$

Note, that in (3.21) we are concentrating on a basis of $(N-3)!$ heterotic subamplitudes referring to the color ordering $(1, \rho(2), \dots, N-3, N-1, N)$. All other heterotic subamplitudes can be expressed in terms of this basis as a result of partial fraction decomposition (cf. Eq. (3.18)) and partial integrations in the left-moving sector. In these lines by using partial fraction decompositions and partial integrations in the left-moving sector we can express the complex integral (3.22) as

$$J[\rho | \bar{\rho}] = \sum_{\tau \in \mathcal{S}_{N-3}} (K^{-1})_{\rho}^{\tau} I[\tau | \bar{\rho}], \quad (3.23)$$

with:

$$\begin{aligned} I[\rho | \bar{\rho}] &:= \left(\prod_{j=2}^{N-2} \int_{z_j \in \mathbb{C}} d^2 z_j \right) \prod_{i < j}^{N-1} |z_{ij}|^{2s_{ij}} \\ &\quad \times \frac{1}{z_{1\rho(2)} z_{\rho(2),\rho(3)} \dots z_{\rho(N-3),\rho(N-2)} \bar{z}_{1\bar{\rho}(2)} \bar{z}_{\bar{\rho}(2),\bar{\rho}(3)} \dots \bar{z}_{\bar{\rho}(N-3),\bar{\rho}(N-2)}}. \end{aligned} \quad (3.24)$$

In (3.24) we have chosen the gauge choice $z_1 = 0$, $z_{N-1} = 1$, $z_N = \infty$ and the remaining $N-3$ positions z_i are integrated on the whole complex sphere. The matrix K_{ρ}^{σ} accounts for the basis change of SYM subamplitudes

$$\tilde{A}(\rho) = K_{\rho}^{\sigma} A(\sigma) \quad (3.25)$$

expressing the basis of $(N-3)!$ SYM subamplitudes

$$\tilde{A}(\rho) := A(1, \rho(2), \dots, N-2, N, N-1) \quad (3.26)$$

in terms of the basis (3.4). According to the arguments given in [20,21], the inverse transposed matrix $K^* := (K^{-1})^t$ describes the corresponding basis change on the open string world-sheet

integrand of the left-moving sector, *i.e.* $Z_\pi(1, \rho(2, \dots, N-2), N-1, N) = Z_\pi(1, \tau(2, \dots, N-2), N, N-1)(K^*)^\tau_\rho$. E.g. for $N=4$ we have $K = \frac{s_{23}}{s_{13}}$, while for $N=5$ the matrix:

$$K_\rho^\sigma = \begin{pmatrix} \frac{(s_{13}+s_{15})s_{34}}{s_{14}s_{35}} & -\frac{s_{13}s_{24}}{s_{14}s_{35}} \\ -\frac{s_{12}s_{34}}{s_{14}s_{25}} & \frac{(s_{12}+s_{15})s_{24}}{s_{14}s_{25}} \end{pmatrix}. \quad (3.27)$$

For $N=6$ the first row of K_ρ^σ reads

$$K_1^\sigma = [s_{15}s_{46}(s_{34} + s_{46} + s_{36})]^{-1} \{s_{45}(s_{13} + s_{14} + s_{16})(s_{26} + s_{36} + s_{56} + s_{35} + s_{61}), \\ -s_{13}s_{25}s_{45}, s_{14}s_{25}(s_{13} + s_{14} + s_{24} + s_{34} + s_{45}), -s_{13}s_{25}(s_{24} + s_{45}), \\ -s_{14}s_{35}(s_{13} + s_{14} + s_{16}), s_{35}(s_{13} + s_{14} + s_{16})(s_{34} + s_{45} + s_{46})\}, \quad (3.28)$$

with the entry $\sigma \in S_3$ referring to the permutations $(2, 3, 4)$, $(3, 2, 4)$, $(4, 3, 2)$, $(3, 4, 2)$, $(4, 2, 3)$ and $(2, 4, 3)$, respectively. The remaining entries of K may be obtained from (3.28) by permuting the numbers 2, 3 and 4 and changing the positions σ in accord with the basis $A(\sigma)$ they refer to. A general formula for K can be derived by rewriting a similar formula given in [4]. After some adjustments we find (for $N \geq 4$)

$$K_1^\sigma = \prod_{l=2}^{N-2} \frac{c(\{1, \sigma(2, \dots, N-2), N-1\}; l)}{(k_N + k_{N-2} + \dots + k_l)^2}, \quad (3.29)$$

with the functions $c = c_1 + c_2$

$$c_1(\{1, \sigma(2, \dots, N-2), N-1\}; l) = \begin{cases} \sum_{o=\tau_l}^{N-1} \kappa_{l, \rho_o}, & \tau_{l+1} < \tau_l, \\ -\sum_{o=1}^{\tau_l} \kappa_{l, \rho_o}, & \tau_{l+1} > \tau_l, \end{cases} \\ c_2(\{1, \sigma(2, \dots, N-2), N-1\}; l) = \begin{cases} (k_N + k_{N-2} + \dots + k_l)^2, & \tau_{l+1} < \tau_l < \tau_{l-1}, \\ -(k_N + k_{N-2} + \dots + k_l)^2, & \tau_{l+1} > \tau_l > \tau_{l-1}, \\ 0 & \text{else,} \end{cases} \quad (3.30)$$

associated to leg l . Above τ_l (with $\tau_1 := 0$ and $\tau_{N-1} := \tau_{N-3}$) is the position of leg l in the set $\rho := \{\rho_1, \dots, \rho_{N-1}\} = \{1, \sigma(2, \dots, N-2), N-1\}$, and:

$$\kappa_{ij} = \begin{cases} s_{ij} & i > j, \text{ or } j = N-1, \\ 0 & \text{else.} \end{cases} \quad (3.31)$$

Finally, with (3.23) we can write (3.21) in matrix notation as

$$\mathcal{A}^{\text{HET}} = (-1)^{N-3} J S A = (-1)^{N-3} K^{-1} I S A, \quad (3.32)$$

with the vector \mathcal{A}^{HET} , whose entries $\mathcal{A}_\rho^{\text{HET}} = \mathcal{A}^{\text{HET}}(\rho)$ are the $(N-3)!$ heterotic subamplitudes (3.21). With the identity $I = K \text{sv}(Z)$, which we will prove in Eq. (4.10), the relations $F = (-1)^{N-3} Z S$ and $\mathcal{A}^I = F A$ following from Eqs. (3.6) and (3.3), respectively we finally have⁶:

$$\mathcal{A}^{\text{HET}} = \text{sv}(\mathcal{A}^I). \quad (3.33)$$

⁶ Note, that with the comments from Footnote 5, the map (2.3) has to be accompanied by the rescaling of the inverse string tension $\alpha' \rightarrow \frac{\alpha'}{4}$.

To conclude, the single trace heterotic gauge amplitudes $\mathcal{A}^{\text{HET}}(\rho)$ referring to the color ordering ρ are simply obtained from the relevant open string gauge amplitudes $\mathcal{A}^{\text{I}}(\rho)$ by imposing the projection sv introduced in (2.3). Therefore, the α' -expansion of the heterotic amplitude $\mathcal{A}^{\text{HET}}(\rho)$ can be obtained from that of the open superstring amplitude $\mathcal{A}^{\text{I}}(\rho)$ by simply replacing MZVs by their corresponding SVMZVs according to the rules (2.4) introduced in (2.3).

As corollary let us recall the relations for the color ordered open superstring subamplitudes

$$\begin{aligned} &\mathcal{A}^{\text{I}}(1, 2, \dots, N) + e^{i\pi s_{12}} \mathcal{A}^{\text{I}}(2, 1, 3, \dots, N-1, N) \\ &+ e^{i\pi(s_{12}+s_{13})} \mathcal{A}^{\text{I}}(2, 3, 1, \dots, N-1, N) \\ &+ \dots + e^{i\pi(s_{12}+s_{13}+\dots+s_{1,N-1})} \mathcal{A}^{\text{I}}(2, 3, \dots, N-1, 1, N) = 0, \end{aligned} \quad (3.34)$$

and permutations thereof following from considering monodromies on the open string world-sheet [2,27]. Applying the map (2.3) on (3.34), thereby using $\text{sv}(\pi) = 0$ and applying (3.33) gives the set of KK equations

$$\begin{aligned} &\mathcal{A}^{\text{HET}}(1, 2, \dots, N) + \mathcal{A}^{\text{HET}}(2, 1, 3, \dots, N-1, N) + \mathcal{A}^{\text{HET}}(2, 3, 1, \dots, N-1, N) \\ &+ \dots + \mathcal{A}^{\text{HET}}(2, 3, \dots, N-1, 1, N) = 0, \end{aligned} \quad (3.35)$$

and the BCJ relations

$$\begin{aligned} &s_{12} \mathcal{A}^{\text{HET}}(2, 1, 3, \dots, N-1, N) + (s_{12} + s_{13}) \mathcal{A}^{\text{HET}}(2, 3, 1, \dots, N-1, N) \\ &+ \dots + (s_{12} + s_{13} + \dots + s_{1,N-1}) \mathcal{A}^{\text{HET}}(2, 3, \dots, N-1, 1, N) = 0, \end{aligned} \quad (3.36)$$

for the heterotic subamplitudes, respectively in agreement with our comments after Eq. (3.22). To conclude, heterotic string subamplitudes fulfill the same amplitude relations, *i.e.* KK and BCJ relations as field-theory amplitudes.

4. Representation of gravitational amplitudes in superstring theory

In this section we shall elaborate on the graviton tree-level scattering amplitude in superstring theory and relate it to the heterotic gauge amplitude computed in the previous section. Furthermore, we will discuss type II and heterotic scalar amplitudes and related them to type I scalar amplitudes.

Thanks to the KLT relations [1] at tree-level closed string amplitudes can be expressed as sum over squares of (color ordered) open string subamplitudes arising from the left- and right-moving sectors. This map gives a relation between a closed string tree-level amplitude involving N closed strings and a sum of squares of (partial ordered) open string tree-level amplitudes each involving N open strings. E.g. for the N -graviton scattering amplitude \mathcal{M} in (type I or type II) superstring theory we may write these identities as follows [1,22,23]

$$\begin{aligned} \mathcal{M}(1, \dots, N) &= (-1)^{N-3} \kappa^{N-2} \sum_{\sigma \in S_{N-3}} \mathcal{A}(1, \sigma(2, 3, \dots, N-2), N-1, N) \\ &\times \sum_{\rho \in S_{N-3}} \mathcal{S}[\rho|\sigma] \tilde{\mathcal{A}}(1, \rho(2, 3, \dots, N-2), N, N-1), \end{aligned} \quad (4.1)$$

with the gravitational coupling constant κ and a product of sin-factors

$$\begin{aligned} \mathcal{S}[\rho|\sigma] &:= \mathcal{S}[\rho(2, \dots, N-2) | \sigma(2, \dots, N-2)] \\ &= \prod_{j=2}^{N-2} \sin \left(s_{1,j_\rho} + \sum_{k=2}^{j-1} \theta(j_\rho, k_\rho) s_{j_\rho, k_\rho} \right), \end{aligned} \quad (4.2)$$

which depend on the kinematic invariants (3.8) and arise from the KLT relations [1,23]. In (4.2) we use the same notation as described below Eq. (3.7). In (4.1) the graviton amplitude is expressed as a sum over $[(N-3)!]^2$ terms each contributing a product of two full-fledged open superstring amplitudes \mathcal{A} and $\tilde{\mathcal{A}}$. The expression (4.1) is a very intricate way of writing a closed string amplitude \mathcal{M} . In fact, in [3] it has already been anticipated, that there are more efficient and elegant ways in writing (4.1) in terms of a sum involving linearly $(N-3)!$ single full-fledged open superstring amplitudes \mathcal{A} only.

Building up on the open superstring result (3.3) the N -graviton amplitude in superstring theory can be written as

$$\mathcal{M}(1, \dots, N) = \kappa^{N-2} \sum_{\sigma \in S_{N-3}} \sum_{\bar{\sigma} \in S_{N-3}} A(\sigma) G[\sigma | \bar{\sigma}] A(\bar{\sigma}), \quad (4.3)$$

with the gravity kernel

$$\begin{aligned} G[\sigma | \bar{\sigma}] &:= G[\sigma(2, 3, \dots, N-2) | \bar{\sigma}(2, 3, \dots, N-2)] \\ &= \sum_{\rho \in S_{N-3}} \sum_{\bar{\rho} \in S_{N-3}} S[\rho | \sigma] I[\rho | \bar{\rho}] S[\bar{\rho} | \bar{\sigma}], \end{aligned} \quad (4.4)$$

and the world-sheet sphere integrals introduced in Eq. (3.24). The field-theory limit of the graviton amplitude (4.3) can be written as

$$\mathcal{M}_{\text{FT}}(1, \dots, N) = (-1)^{N-3} \kappa^{N-2} \sum_{\sigma \in S_{N-3}} \sum_{\bar{\sigma} \in S_{N-3}} A(\sigma) S_0[\sigma | \bar{\sigma}] A(\bar{\sigma}), \quad (4.5)$$

with the intersection matrix S_0 whose entries are $S_{0, \rho, \sigma} := S_0[\rho | \sigma]$. The limit (4.5) has to agree with the expression (with $\tilde{\mathcal{A}}$ defined in (3.26))

$$\mathcal{M}_{\text{FT}}(1, \dots, N) = (-1)^{N-3} \kappa^{N-2} \sum_{\sigma \in S_{N-3}} \sum_{\bar{\sigma} \in S_{N-3}} A(\sigma) S[\sigma | \bar{\sigma}] \tilde{\mathcal{A}}(\bar{\sigma}), \quad (4.6)$$

which directly follows from (4.1). Comparing the two expressions (4.5) and (4.6) and using the basis change (3.25) yields:

$$S_0 = SK. \quad (4.7)$$

One key observation in [3] is that the closed superstring amplitude (4.3) can be expressed in terms of a linear combination of $(N-3)!$ open superstring amplitudes (3.3) as

$$\mathcal{M} = (-1)^{N-3} \kappa^{N-2} A^t S_0 \text{sv}(\mathcal{A}^{\text{I}}), \quad (4.8)$$

subject to the map sv introduced in (2.3) and the intersection matrix S_0 describing the field-theory limit (4.5) of the graviton amplitude (4.3). As a consequence of (4.8) for the gravity kernel (4.4) we have

$$G = S^t I S = (-1)^{N-3} S_0 \text{sv}(F), \quad (4.9)$$

i.e. with $F = (-1)^{N-3} ZS$ from (3.6) and (4.7) we obtain:

$$I = K \operatorname{sv}(Z). \quad (4.10)$$

With our result (3.33) we can write the tree-level graviton N -point amplitude (4.8) of superstring theory as

$$\mathcal{M} = (-1)^{N-3} \kappa^{N-2} A^t S_0 \mathcal{A}^{\text{HET}}, \quad (4.11)$$

with the vector \mathcal{A}^{HET} of $(N-3)!$ heterotic single trace tree-level gauge N -point amplitudes (3.33). Note, that by applying naively KLT relations we would not have arrived at (4.11). The relation (4.11) connects two seemingly different amplitudes from two different string vacua. This might provide an extension to the heterotic–type I duality [28], see also Section 6.

Obviously, in four-dimensional $\mathcal{N} = 8$ type II superstring vacua the relationship (4.8) can be applied for closed string amplitudes involving scalars. The latter belong to the supergravity multiplet and a subclass Φ^{ij} , $i, j = 1, \dots, 6$, of them denote geometric moduli fields describing the internal $D = 6$ toroidal compactification with the metric g^{ij} . For instance, the four-scalar amplitude in four-dimensional $\mathcal{N} = 8$ type II string vacua reads

$$\begin{aligned} \mathcal{A}^{\text{II}}(\Phi_1^{i_1 j_1}, \Phi_2^{i_2 j_2}, \Phi_3^{i_3 j_3}, \Phi_4^{i_4 j_4}) &= \frac{u}{st} \left(t\delta_1 + s\delta_2 + \frac{st}{u}\delta_3 \right) \left(t\bar{\delta}_1 + s\bar{\delta}_2 + \frac{st}{u}\bar{\delta}_3 \right) \\ &\times \frac{\Gamma(s)\Gamma(u)\Gamma(t)}{\Gamma(-s)\Gamma(-u)\Gamma(-t)}, \end{aligned} \quad (4.12)$$

with:

$$\begin{aligned} \delta_1 &= g^{i_1 i_2} g^{i_3 i_4}, & \delta_2 &= g^{i_1 i_3} g^{i_2 i_4}, & \delta_3 &= g^{i_1 i_4} g^{i_2 i_3}, \\ \bar{\delta}_1 &= g^{j_1 j_2} g^{j_3 j_4}, & \bar{\delta}_2 &= g^{j_1 j_3} g^{j_2 j_4}, & \bar{\delta}_3 &= g^{j_1 j_4} g^{j_2 j_3}. \end{aligned} \quad (4.13)$$

The kinematic invariants (3.8) are given by $s = \alpha'(k_1 + k_2)^2$, $t = \alpha'(k_1 + k_3)^2$, $u = \alpha'(k_1 + k_4)^2$, with $s + t + u = 0$. On the other hand, in four-dimensional $\mathcal{N} = 4$ type I string vacua the four-point subamplitude of scalar fields Φ^j , $j = 1, \dots, 6$, describing open string moduli (transversal D -brane positions or Wilson lines) reads

$$\mathcal{A}^{\text{I}}(\Phi_1^{j_1}, \Phi_2^{j_2}, \Phi_3^{j_3}, \Phi_4^{j_4}) = \left(t\bar{\delta}_1 + s\bar{\delta}_2 + \frac{st}{u}\bar{\delta}_3 \right) \frac{\Gamma(s)\Gamma(1+u)}{\Gamma(1+s+u)}, \quad (4.14)$$

w.r.t. to the color ordering (1, 2, 3, 4) and the symbols defined in (4.13). Comparing the two amplitudes (4.12) and (4.14) and using Eq. (5.3) yields the correspondence

$$\begin{aligned} \mathcal{A}^{\text{II}}(\Phi_1^{i_1 j_1}, \Phi_2^{i_2 j_2}, \Phi_3^{i_3 j_3}, \Phi_4^{i_4 j_4}) &= -\frac{u}{t} \left(t\delta_1 + s\delta_2 + \frac{st}{u}\delta_3 \right) \\ &\times \operatorname{sv}(\mathcal{A}^{\text{I}}(\Phi_1^{j_1}, \Phi_2^{j_2}, \Phi_3^{j_3}, \Phi_4^{j_4})), \end{aligned} \quad (4.15)$$

in lines of (4.8). Similar relation than (4.15) can also be derived for amplitudes involving more than four scalar fields.

Finally, let us briefly discuss the heterotic analog of the scalar amplitude (4.12). The four-scalar amplitude in four-dimensional $\mathcal{N} = 4$ heterotic string vacua reads

$$\begin{aligned} \mathcal{A}^{\text{HET}}(\Phi_1^{i_1 j_1}, \Phi_2^{i_2 j_2}, \Phi_3^{i_3 j_3}, \Phi_4^{i_4 j_4}) &= \frac{u}{st} \left(\frac{t}{1-s}\delta_1 + \frac{s}{1-t}\delta_2 + \frac{t}{1-u}\frac{s}{u}\delta_3 \right) \\ &\times \left(t\bar{\delta}_1 + s\bar{\delta}_2 + \frac{st}{u}\bar{\delta}_3 \right) \frac{\Gamma(s)\Gamma(u)\Gamma(t)}{\Gamma(-s)\Gamma(-u)\Gamma(-t)}, \end{aligned} \quad (4.16)$$

with (4.13). An expression similar to (4.16) can be derived for the four-scalar amplitude involving scalars, which describe Wilson line moduli. In four-dimensional $\mathcal{N} = 4$ heterotic string theory the scalars Φ^{ij} belong to vector multiplets. In (4.16) the spurious tachyonic poles, which arise from the exchange of fields from the massless $\mathcal{N} = 4$ supergravity multiplet, have no counterpart in the tree-level perturbative type I superstring amplitude (4.14). Nevertheless, the full α' -dependence of the heterotic amplitude (4.16) can again be described by the open string amplitude (4.14). Comparing the two amplitudes (4.16) and (4.14) and using Eq. (5.3) yields:

$$\mathcal{A}^{\text{HET}}(\Phi_1^{i_1 j_1}, \Phi_2^{i_2 j_2}, \Phi_3^{i_3 j_3}, \Phi_4^{i_4 j_4}) = -\frac{u}{t} \left(\frac{t}{1-s} \delta_1 + \frac{s}{1-t} \delta_2 + \frac{t}{1-u} \frac{s}{u} \delta_3 \right) \times \text{sv}(\mathcal{A}^1(\Phi_1^{j_1}, \Phi_2^{j_2}, \Phi_3^{j_3}, \Phi_4^{j_4})). \quad (4.17)$$

Amplitudes involving more than four scalar fields can be cast into a similar form than (4.17). Note, that by applying naively KLT relations we would not have arrived at (4.17). Connections in lines of (4.17) can also be established for heterotic gravitational amplitudes or non-single trace gauge amplitudes subject to gravitational exchanges.

The lesson to learn from the example (4.17) is that also closed string amplitudes other than heterotic (single-trace) gauge (3.33) or superstring gravitational amplitudes (4.8) can be expressed as single-valued image of some open string amplitudes. Generically, after performing partial fraction decompositions and partial integration relations any complex integral referring to a closed string world-sheet integral can be expressed in terms of the fundamental basis (3.22), which serves as building block for complex integrals. The elements of this basis can be written as single-valued image of some open string world-sheet integrals (*cf.* Section 5). Therefore, any closed string amplitude can be written as single-valued image of open string amplitudes by expressing the underlying closed string world-sheet integrals as single-valued image of open string integrals. As a consequence the whole α' -dependence of closed string amplitudes is entirely encapsulated in the corresponding open string amplitude.

5. Complex vs. iterated integrals: closed vs. open string world-sheet integrals

Open string world-sheet disk integrals (3.5) are described as iterated (real) integrals on $\mathbf{RP}^1 \setminus \{0, 1, \infty\}$, while closed string world-sheet sphere integrals (3.22) are given by integrals over the full complex plane. The latter, which can be considered as iterated integrals on $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ integrated independently on all choices of paths, are more involved than the real iterated integrals appearing in open string amplitudes. Nevertheless, in the previous two sections we have exhibited non-trivial relations between open and closed string amplitudes and in this section we shall elaborate on these connections at the level of the world-sheet integrals. In this section we shall show, that quite generally complex integrals can be expressed as real iterated integrals subject to the projection sv .

Recall, that Eq. (4.10) expresses complex sphere integrals (3.24) in terms of a linear combination of disk integrals (3.5) subject to the map sv . This is to be contrasted with the KLT formula (4.1), where squares of disk integrals (3.5) appear. In light of (4.10) let us discuss the simplest case describing the scattering of four closed strings. For $N = 4$ the real integral (3.5) becomes

$$Z := Z_1(1) = - \int_0^1 dx x^{s-1} (1-x)^u = - \frac{\Gamma(s) \Gamma(1+u)}{\Gamma(1+s+u)}, \quad (5.1)$$

while the complex integral (3.24) boils down to:

$$I := I[1|1] = \int_{\mathbf{C}} d^2 z |z|^{2s-2} |1-z|^{2u} = \frac{u}{st} \frac{\Gamma(s)\Gamma(u)\Gamma(t)}{\Gamma(-s)\Gamma(-u)\Gamma(-t)}. \quad (5.2)$$

With $K = \frac{u}{t}$ Eq. (4.10) gives rise to:

$$\frac{u}{st} \frac{\Gamma(s)\Gamma(u)\Gamma(t)}{\Gamma(-s)\Gamma(-u)\Gamma(-t)} = -\frac{u}{t} \text{sv} \left(\frac{\Gamma(s)\Gamma(1+u)}{\Gamma(1+s+u)} \right), \quad (5.3)$$

i.e.:

$$\int_{\mathbf{C}} d^2 z |z|^{2s-2} |1-z|^{2u} = -\frac{u}{t} \text{sv} \left(\int_0^1 dx x^{s-1} (1-x)^u \right). \quad (5.4)$$

Similar explicit correspondences (4.10) between complex sphere integrals I and real disk integrals Z can be made for $N \geq 5$. Finally, let us note, that with (3.9) the complex integrals (3.24) have the following field-theory limit:

$$I|_{\alpha'=0} = (-1)^{N-3} K S^{-1}. \quad (5.5)$$

Moreover, a direct correspondence between complex sphere integrals and real disk integrals can be made for the (heterotic) world-sheet integrals (3.22). Indeed, with (3.23), i.e. $I = KJ$ Eq. (4.10) becomes:

$$J = \text{sv}(Z). \quad (5.6)$$

As one implication of (5.6) to each *single* complex sphere integral $J[\pi|\rho]$ one real integral $Z_\pi(\rho)$ corresponds. For our $N = 4$ example we now have:

$$J := J[1|1] = - \int_{\mathbf{C}} d^2 z |z|^{2s-2} |1-z|^{2u} (1-z)^{-1} = \frac{1}{s} \frac{\Gamma(s)\Gamma(u)\Gamma(t)}{\Gamma(-s)\Gamma(-u)\Gamma(-t)}. \quad (5.7)$$

With (5.1) Eq. (5.6) gives rise to:

$$\int_{\mathbf{C}} d^2 z \frac{|z|^{2s} |1-z|^{2u}}{z(1-z)\bar{z}} = \text{sv} \left(\int_0^1 dx x^{s-1} (1-x)^u \right). \quad (5.8)$$

Hence, in (3.22) the effect of inserting the left-moving gauge part (3.17) is simply the projection (2.3) acting on the right-moving part (3.5). Similar explicit and direct correspondences (5.6) between the complex sphere integrals Z and the real disk integrals Z can be made for $N \geq 5$. In order to familiarize with the matrix notation let us explicitly write the case (5.6) for $N = 5$ (with $z_1 = 0, z_4 = 1$):

$$\begin{aligned} & \left(\int_{z_2, z_3 \in \mathbf{C}} d^2 z_2 d^2 z_3 \frac{\prod_{i < j}^4 |z_{ij}|^{2s_{ij}}}{z_{12} z_{23} z_{34} \bar{z}_{12} \bar{z}_{23}} \int_{z_2, z_3 \in \mathbf{C}} d^2 z_2 d^2 z_3 \frac{\prod_{i < j}^4 |z_{ij}|^{2s_{ij}}}{z_{12} z_{23} z_{34} \bar{z}_{13} \bar{z}_{32}} \right) \\ & \left(\int_{z_2, z_3 \in \mathbf{C}} d^2 z_2 d^2 z_3 \frac{\prod_{i < j}^4 |z_{ij}|^{2s_{ij}}}{z_{13} z_{32} z_{24} \bar{z}_{12} \bar{z}_{23}} \int_{z_2, z_3 \in \mathbf{C}} d^2 z_2 d^2 z_3 \frac{\prod_{i < j}^4 |z_{ij}|^{2s_{ij}}}{z_{13} z_{32} z_{24} \bar{z}_{13} \bar{z}_{32}} \right) \\ & = \text{sv} \left(\int_{0 < z_2 < z_3 < 1} dz_2 dz_3 \frac{\prod_{i < j}^4 |z_{ij}|^{s_{ij}}}{z_{12} z_{23}} \int_{0 < z_2 < z_3 < 1} dz_2 dz_3 \frac{\prod_{i < j}^4 |z_{ij}|^{s_{ij}}}{z_{13} z_{32}} \right. \\ & \quad \left. \int_{0 < z_3 < z_2 < 1} dz_2 dz_3 \frac{\prod_{i < j}^4 |z_{ij}|^{s_{ij}}}{z_{12} z_{23}} \int_{0 < z_3 < z_2 < 1} dz_2 dz_3 \frac{\prod_{i < j}^4 |z_{ij}|^{s_{ij}}}{z_{13} z_{32}} \right). \end{aligned} \quad (5.9)$$

In (5.9) we explicitly see how the presence of the left-moving gauge insertion in the complex integrals results in the projection onto real integrals involving only the right-moving part. Besides, let us compute the closed string analog of (3.9). With (5.6) and (3.9) we find:

$$J|_{\alpha'=0} = (-1)^{N-3} S^{-1}. \quad (5.10)$$

Hence, the set of complex world-sheet sphere integrals (3.22) are the closed string analogs of the open string world-sheet disk integrals (3.5).

To conclude, after applying partial integrations to remove double poles, which are responsible for spurious tachyonic poles, all closed superstring amplitudes can be expressed in terms of the basis (3.22), which in turn through (5.6) can be related to the basis of open string amplitudes (3.5). As a consequence the α' -dependence of any closed string amplitude is given by that of the underlying open string amplitudes, cf. Eqs. (3.33), (4.8) and (4.17) as some examples.

Finally, we would like to make a connection to Ref. [8]. In this reference it has been argued, that the field-theory limit (3.9) of the open string world-sheet disk integrals (3.5) is related to the double partial amplitudes of a massless colored cubic scalar theory

$$m_N^{(0)}(\alpha|\beta) = \int \frac{d^n \sigma}{\text{Vol}(\text{SL}(2, \mathbb{C}))} \frac{\prod_a' \delta(\sum_{b \neq a} \frac{s_{ab}}{\sigma_{ab}})}{(\sigma_{\alpha(1), \alpha(2)} \cdots \sigma_{\alpha(N), \alpha(1)}) (\sigma_{\beta(1), \beta(2)} \cdots \sigma_{\beta(N), \beta(1)})}, \quad (5.11)$$

evaluated at the solutions of the scattering equations $\sum_{b \neq a} \frac{s_{ab}}{\sigma_{ab}} = 0$. More precisely, with $(m_{\text{scalar}})_{\alpha\beta} := m_N^{(0)}(1, \alpha(2), \dots, N-2, N-1, N|1, \beta(2), \dots, N-2, N, N-1)$ we have [8]:

$$(m_{\text{scalar}})_{\alpha\beta} = Z|_{\alpha'=0} = (-1)^{N-3} (S^{-1})_{\alpha\beta}. \quad (5.12)$$

With (5.10) we now also obtain a relation to the building blocks (3.22) of the heterotic string amplitudes as:

$$(m_{\text{scalar}})_{\alpha\beta} = J|_{\alpha'=0} = (-1)^{N-3} (S^{-1})_{\alpha\beta}. \quad (5.13)$$

It is quite striking, that the structure of (5.11) and (5.12), furnished by the permutations α and β , is captured by the left- and right-moving parts in (3.22), respectively. Recently in Ref. [29], a similar observation has been made in the heterotic version of Berkovits new twistor-like superstring theory [30]. It would be very interesting to find further connections between the work [8] (and also [29]) and perturbative heterotic string amplitudes presented here.

6. Unity of tree-level superstring amplitudes

It is well known that various formulations/compactifications of superstring theory are connected by a web of dualities. They can be interpreted as different vacua of a universal M-theory [31]. The classic example is type IIA/K3–heterotic/ T^4 duality in six dimensions [32]. Similar to many other examples, it is a strong–weak coupling duality. Perturbative states on one side, like heterotic gauge bosons, are mapped to non-perturbative states on the other side, like D-branes wrapping on K3 cycles. There are convincing arguments, in all known duality cases, that such correspondence holds at the massless level. It is not clear, however, what is the role of Regge excitations in strong–weak coupling dualities. It is regrettable because, without exaggerating, the Regge states, as arising from string vibrations, are the true essence of string theory. The α' -dependence of the amplitudes discussed in this paper are due to such Regge states propagating in all possible channels.

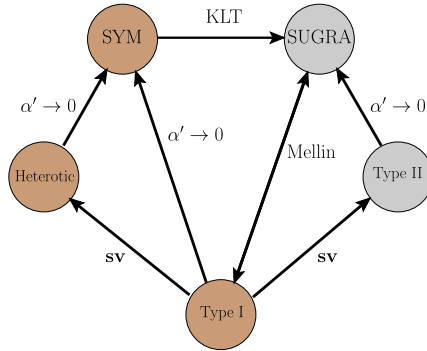


Fig. 1. Unity of tree-level superstring amplitudes.

One notable exception is type I–heterotic duality in four dimensions [28]. There is a class of effective action terms, essentially describing the (generalized) non-Abelian Born–Infeld action, which appears at the tree-level on type I side, while it is generated by loop corrections on the heterotic side [33]. Massive string excitations appear to play some role in this correspondence because on type I side, the Born–Infeld terms appear in the α' -loop expansion of the two-dimensional world-sheet sigma model, while in space–time, the corresponding interactions are mediated by Regge states at the tree level. The comparison with heterotic theory works well at the one-loop level $\mathcal{O}(F^4)$, but runs into problems at two loops $\mathcal{O}(F^6)$ [34].

The single-valued projection connecting type I and heterotic single-trace amplitudes creates a new link in the web of relations shown in Fig. 1. Although similar to the duality web, the “web of amplitudes” links the scattering amplitudes evaluated (to all orders in α') not only in different string vacua but, what is most important, it also includes some links between the amplitudes involving external particles *not* related by supersymmetry or any known symmetry, like gravitons and gauge bosons.

Type I open string theory appears to play a central role in the web of amplitudes. By single-valued projections, it generates single-trace heterotic gauge amplitudes (3.33) and type II graviton amplitudes (4.8) [3]. E.g. (4.11) provides though a bridge from gauge heterotic to type II graviton amplitude bypassing type I, without using sv projections. Type I connects also, via Mellin correspondence, to $\mathcal{N} = 8$ supergravity [9]. On the other hand, KLT allows constructing supergravity amplitudes from $\mathcal{N} = 4$ SYM which supplies the basic building blocks for all tree-level open and closed string amplitudes.

It should be made clear that the connections depicted in Fig. 1 appear in perturbation theory, at the tree-level, in four space–time dimensions. The form of vertex operators is determined by world-sheet supersymmetry, but their world-sheet correlators are decoupled from the internal sector of SCFT associated to compact dimensions. In order to see the effects of internal dimensions, one would have to consider the amplitudes involving moduli fields like in Eq. (4.17); one could also go beyond the tree-level to observe internal states and their Regge excitations propagating in the loops. One of the nodes obviously missing in Fig. 1 are the heterotic amplitudes involving external gravitons, which must be sensitive to the massive string spectrum with a lower, $\mathcal{N} = 4$ supersymmetry, as compared to $\mathcal{N} = 8$ of type II, *cf.* also the comments at the end of Section 4.

There must be a deep reason for the universal α' dependence of all tree-level string amplitudes and their connection to SYM and supergravity. It is possible that some new insights can be gained by connecting the web of amplitudes with string dualities.

7. Concluding remarks

In this work we have found a correspondence between closed and open superstring amplitudes communicated by the sv map (2.3). This map relates two string amplitudes of different world-sheet topologies. One basic example is the relation (3.33) between the single trace heterotic tree-level gauge amplitudes (3.20) and open superstring tree-level gauge amplitudes (3.1). Based on this example many other closed/open amplitude connections can be established, e.g. relations (4.15) or (4.17) between type II or heterotic and type I scalar amplitudes, respectively. The essential property common to all such relationships is that the full α' -dependence of the type II or heterotic closed string amplitudes is encapsulated in the type I open string amplitudes. These relations give rise to a much deeper connection between open and closed string amplitudes than what is implied by KLT relations. It would be interesting to understand the sv map (2.3) in the framework of sigma-model expansion in the underlying superconformal world-sheet theory. Also important is to clarify the role of the map (2.3) at the level of perturbation theory of open and closed strings from the nature of their underlying string world-sheets.

Furthermore, we have established a connection between the single trace heterotic tree-level gauge amplitudes and graviton amplitudes of superstring theory, cf. (4.11). This result is quite surprising because it relates gauge and gravitational amplitudes in two different string vacua. On the other hand, in four space–time dimensions one has the relation [35,36]

$$\Delta_{E_6} - \Delta_{E_8} = 12F_1, \quad (7.1)$$

which connects type II one-loop superstring corrections to R^2 expressed by the topological one-loop partition function F_1 (which in turn is related to the generalized $\mathcal{N} = 2$ index) to a difference of one-loop gauge corrections of heterotic (2, 2) Calabi–Yau vacua. In Ref. [36] the relation (7.1) is explained at the level of the underlying world-sheet superconformal field theory as a consequence of the bosonic/supersymmetric map [37]. It is possible that the sv map is connected to such a bosonic/supersymmetric map.

The growing set of interconnections hints towards a fascinating unity of closed and open string amplitudes with gauge theory and supergravity.

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